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Duality transformations for spin-$\frac{1}{2}$ lattice systems

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Abstract. By using a procedure introduced in a previous work, we consider duality transformations for spin-$\frac{1}{2}$ lattice systems. We show that, in some cases, given a self-dual model, a new self-dual model can be obtained from it. Several examples of this fact are given. In particular we analyse a system consisting of two interacting Baxter-Wu models, which presents close analogies with the Ashkin-Teller model.

1. Introduction

In a previous paper (Giacomini 1985) we introduced a straightforward algebraic procedure to obtain duality relations in spin-$\frac{1}{2}$ lattice systems. The simplicity of the procedure enables us to analyse the possibility of generalising a self-dual model in such a way that the new one is self-dual too. Following this idea we consider the Baxter-Wu model (Baxter and Wu 1973) and show that it is possible to incorporate a two-spin interaction without losing the self-duality condition.

Another model that we analyse is defined on a two-dimensional triangular lattice and contains an external field and an alternate three-spin interaction (Merlini and Gruber 1972). This model is self-dual, but the incorporation of a two-spin interaction only in one direction preserves the self-duality property. If we express the Hamiltonian of these models as a sum over sites of the lattice, we conclude that the term added in each case is the product of two terms of the original Hamiltonian, associated to the same site.

In addition, we introduce a two-dimensional model consisting of two interacting Baxter-Wu models, and show that it is self-dual. This model gives a good analogy with the Ashkin-Teller one (Ashkin and Teller 1943).

Finally, we consider the dual system to the square Ising model with next-nearest-neighbour interactions. This dual model consists of two interacting Ising models and presents a four-spin interaction term. This fact could be the reason for the non-universal behaviour of this model (Barber 1980), as happens for Baxter's model (Baxter 1971) and the Ashkin-Teller model. In the next section we deduce the self-duality of the Baxter-Wu model applying the procedure of our previous paper, with the aim of introducing it.

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2. Self-duality of the Baxter–Wu model

The partition function of this model is given by

$$Z=\sum_{s}\exp\left(\sum_{x}K\left(s_x s_{x+\mu}s_{x+\nu}+s_{x+\mu}s_{x+\nu}s_{x+\mu+\nu}\right)\right) \tag{2.1}$$

with $K=J/kT$, and in which we have considered the triangular lattice as a square one with diagonal bonds, so that we can describe the sites in terms of rectangular coordinates. The term x indicates a generic point of the lattice, and μ and ν are unit vectors in the horizontal and vertical directions, respectively. We assume a lattice with periodic boundary conditions in all cases.

By linearising (2.1) in the spin variables, we obtain the result

$$Z=\cosh(K)^{2N}\sum_{s}\prod_{x}(1+\alpha s_x s_{x+\mu}s_{x+\nu})(1+\alpha s_{x+\mu}s_{x+\nu}s_{x+\mu+\nu}) \tag{2.2}$$

where $\alpha=\tanh(K)$, and N is the number of sites. Following the procedure of our previous paper, we expand the products that appear in (2.2), by introducing two variables n and m, which can take the value zero and one. In this way we obtain

$$Z=\cosh(K)^{2N}\sum_{n,m,s}\prod_{x}\alpha^{n_x+m_x}s_x^{n_x+n_{x-\mu}+n_{x-\nu}+m_{x-\nu}+m_{x-\mu}+m_{x-\mu-\nu}}. \tag{2.3}$$

The spin variable s is uncoupled in (2.3) and can be summed up. Therefore

$$Z=(2\cosh^2(K))^N\sum_{n,m}\prod_{x}\alpha^{n_x+m_x}\delta_2(n_x+n_{x-\mu}+n_{x-\nu}+m_{x-\mu}+m_{x-\nu}+m_{x-\mu-\nu}) \tag{2.4}$$

where $\delta_2(n)$ is a Kronecker delta function modulo two, it is zero if n is odd and one if n is even.

By introducing Ising variables as follows: $s_x^{(1)}=2n_x-1$, $s_x^{(2)}=2m_x-1$ the delta function can be expressed as

$$\begin{aligned}\delta_z(n_x+n_{x-\mu}+n_{x-\nu}+m_{x-\mu}+m_{x-\nu}+m_{x-\mu-\nu})\\=\tfrac{1}{2}(1+s_x^{(1)}s_{x-\mu}^{(1)}s_{x-\nu}^{(1)}s_{x-\mu}^{(2)}s_{x-\nu}^{(2)}s_{x-\mu-\nu}^{(2)}).\end{aligned} \tag{2.5}$$

Besides, using the identities

$$\begin{aligned}\alpha^{n_x+m_x}&=[1+n_x(\alpha-1)][1+m_x(\alpha-1)]\\&=\tfrac{1}{4}[\alpha+1+(\alpha-1)s_x^{(1)}][\alpha+1+(\alpha-1)s_x^{(2)}]\end{aligned} \tag{2.6}$$

(2.4) becomes

$$Z=(\tfrac{1}{2}\mathrm{e}^K)^{2N}\sum_{s^{(1)},s^{(2)}}\prod_{x}(1+vs_x^{(1)})(1+vs_x^{(2)})(1+s_x^{(1)}s_{x-\mu}^{(1)}s_{x-\nu}^{(1)}s_{x-\mu}^{(2)}s_{x-\nu}^{(2)}s_{x-\mu-\nu}^{(2)}) \tag{2.7}$$

where $v=(1-\alpha)/(1+\alpha)$.

By making a translation on the lattice according to

$$s_x^{(1)}\to s_{x+\mu+\nu}^{(1)}\qquad s_x^{(2)}\to s_{x+\mu+\nu}^{(2)}$$

we obtain

$$Z=(\tfrac{1}{2}\mathrm{e}^K)^{2N}\sum_{s^{(1)},s^{(2)}}\prod_{x}(1+vs_x^{(1)})(1+vs_x^{(2)})(1+s_x^{(2)}s_{x+\mu}^{(2)}s_{x+\nu}^{(2)}s_{x+\mu}^{(1)}s_{x+\nu}^{(1)}s_{x+\mu+\nu}^{(1)}). \tag{2.8}$$

In these transformations, the periodic boundary conditions that we have imposed on the lattice play an important role. As can be seen in (2.8), we must solve the constraint

$$s^{(1)}_{x+\mu}s^{(1)}_{x+\nu}s^{(1)}_{x+\mu+\nu}s^{(2)}_{x}s^{(2)}_{x+\mu}s^{(2)}_{x+\nu}=1 \tag{2.9}$$

for all sites x on the lattice. With the aim of simplifying this equation we make the following change of variable

$$s^{(2)}_x \to s^{(1)}_x s^{(2)}_x s^{(2)}_{x+\mu+\nu}. \tag{2.10}$$

Therefore (2.9) results

$$s^{(1)}_x s^{(1)}_{x+\mu+\nu}s^{(2)}_{x+\mu}s^{(2)}_{x+\nu}s^{(2)}_{x+\mu+\nu}s^{(2)}_{x+2\mu+\nu}s^{(2)}_{x+\mu+2\nu}s^{(2)}_{x}=1 \tag{2.11}$$

if we call $s^{(3)}_x=s^{(1)}_x s^{(2)}_{x+\mu}s^{(2)}_{x+\nu}s^{(2)}_x$, (2.11) becomes

$$s^{(3)}_x s^{(3)}_{x+\mu+\nu}=1. \tag{2.12}$$

This equation implies that the spins on a given NW–SE diagonal are equal. One solution of this condition is $s^{(3)}=1$ for all x. Choosing a 'boundary condition' that all $s^{(3)}$ spins in the bottom row are equal to $+1$, this would be the only solution. This fact shows that the contribution of the other solutions of (2.12) must vanish in the thermodynamic limit $N\to\infty$. Therefore they will not be considered.

Hence

$$s^{(1)}_x=s^{(2)}_x s^{(2)}_{x+\mu}s^{(2)}_{x+\nu}. \tag{2.13}$$

Taking (2.10) and (2.13) into account, (2.8) gives

$$Z=(\tfrac{1}{2}\,e^{2K})^N\sum_{s^{(2)}}\prod_x(1+vs^{(2)}_x s^{(2)}_{x+\mu}s^{(2)}_{x+\nu})(1+vs^{(2)}_{x+\mu}s^{(2)}_{x+\nu}s^{(2)}_{x+\mu+\nu}). \tag{2.14}$$

This expression can be written as

$$Z=(\tfrac{1}{2}e^{2K}\cosh^{-2}(K^*))^N\sum_s\exp\left(\sum_x K^*(s_x s_{x+\mu}s_{x+\nu}+s_{x+\mu}s_{x+\nu}s_{x+\mu+\nu})\right) \tag{2.15}$$

where $\tanh(K^*)=v$. This is the well known self-duality relation for this model (Wood and Griffiths 1972, Wood and Pegg 1977), which has been derived here in a straightforward fashion.

3. Self-duality of the Baxter–Wu model with a two spin interaction.

Before we consider this model we will set up some basic notation. For each variable $K_i=J_i/kT$, we define a dual variable K_i^* by

$$\exp(-2K_i)=\tanh(K_i^*). \tag{3.1}$$

Besides, given three interaction coefficients K_1, K_2, K_3, we call

$$\varepsilon=(\tfrac{1}{8}\sinh(2K_1)\sinh(2K_2)\sinh(2K_3))^{1/2}. \tag{3.2}$$

From K_1^*, K_2^*, K_3^*, we define R, M_1, M_2, M_3 such that if a, b, c, d are any four spins with values ±1

$$\sum_d\exp[-(K_1^*a+K_2^*b+K_3^*c)d]=R\exp[M_1bc+M_2ac+M_3ab] \tag{3.3}$$

where

$$
\begin{aligned}
e^{4M_1} &= \frac{\cosh(K_1^*+K_2^*+K_3^*)\cosh(-K_1^*+K_2^*+K_3^*)}{\cosh(K_1^*-K_2^*+K_3^*)\cosh(K_1^*+K_2^*-K_3^*)} \\
e^{4M_2} &= \frac{\cosh(K_1^*+K_2^*+K_3^*)\cosh(-K_2^*+K_1^*+K_3^*)}{\cosh(K_2^*-K_1^*+K_3^*)\cosh(K_2^*-K_3^*+K_1^*)} \\
e^{4M_3} &= \frac{\cosh(K_1^*+K_2^*+K_3^*)\cosh(-K_3^*+K_1^*+K_2^*)}{\cosh(K_3^*-K_1^*+K_2^*)\cosh(K_3^*-K_2^*+K_1^*)} \\
R &= 2\exp(-M_1-M_2-M_3)\cosh(K_1^*+K_2^*+K_3^*).
\end{aligned}
\tag{3.4}
$$

These expressions are the algebraic form of the 'star-triangle' relation (Wannier 1945, Baxter 1982), and will be useful in the following. Moreover, for brevity we will set

$$
s_x^{(i)} = s_i, \qquad s_{x+\mu}^{(i)} = s_i', \qquad s_{x+\nu}^{(i)} = s_i'', \qquad s_{x+\mu+\nu}^{(i)} = s_i''' \tag{3.5}
$$

as shown in figure 1. With this notation, the Hamiltonian of the model that we will consider is

$$
H = -\sum_x (J_1 ss's'' + J_2 s's''s''' + J_3 ss'''). \tag{3.6}
$$

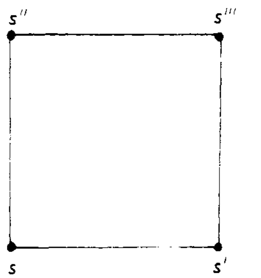

Figure 1. s, s', s'', s''' are four spins round a face of the square lattice.

The new term added is the product of the two terms associated at a given site x in the Baxter-Wu model (we make reference to the spin variables only).

The partition function is given by

$$
Z = \sum_s \exp\left(\sum_x K_1 ss's'' + K_2 s's''s''' + K_3 ss'''\right). \tag{3.7}
$$

By the same procedure of § 2, we obtain

$$
\begin{aligned}
Z = [\tfrac{1}{8}\exp(K_1+K_2+K_3)]^N \sum_{s_1,s_2,s_3} \prod_x (1+v_1 s_1)(1+v_2 s_2)(1+v_3 s_3) \\
\times (1 + s_1' s_1'' s_1''' s_2 s_2' s_2'' s_3 s_3''')
\end{aligned}
\tag{3.8}
$$

where $v_i = \exp(-2K_i)$. Introducing the dual parameters K_i^* (3.8) can be written as

$$
Z = \varepsilon^N \sum_{s_1,s_2,s_3} \prod_x \exp[-(K_1^* s_1 + K_2^* s_2 + K_3^* s_3)](1 + s_1' s_1'' s_1''' s_2 s_2' s_2'' s_3 s_3'''). \tag{3.9}
$$

Now, the variable s_3 can be eliminated. With this purpose, we first change the variables according to $s_1 \to s_1 s_3$, $s_2 \to s_2 s_3$, giving

$$Z = \varepsilon^N \sum_{s_1,s_2,s_3} \prod_x \exp[-(K_1^* s_1 s_3 + K_2^* s_2 s_3 + K_3^* s_3)](1 + s_1' s_1'' s_1''' s_2 s_2' s_2''). \tag{3.10}$$

Since the spin s_3 is decoupled we can sum it for each site. In consequence, using the star-triangle relation we obtain

$$Z = (R\varepsilon)^N \sum_{s_1,s_2} \prod_x \exp(M_1 s_2 + M_2 s_1 + M_3 s_1 s_2)(1 + s_1' s_1'' s_1''' s_2 s_2' s_2''). \tag{3.11}$$

The resulting constraint is equal to (2.9). Hence the partition function results

$$Z = (2R\varepsilon)^N \sum_s \exp\left(\sum_x M_1 s' s'' s''' + M_2 s s' s'' + M_3 s s'''\right). \tag{3.12}$$

Therefore the model is self-dual. Besides, the structure of the self-duality relation is the same as the anisotropic triangular lattice Ising model and a model with four and three spin interactions, analysed in a previous paper (Giacomini 1985).

It is interesting to point out that the change of variables that enables us to decouple the spin s_3, results from the fact that the new term added to the Baxter–Wu model is the product of two terms of the original Hamiltonian, associated to the same site x of the lattice. This fact will be also present in the model treated in the next section.

4. Self-duality of a model with two and three-spin interactions

Let us consider a model defined on a two-dimensional triangular lattice, whose Hamiltonian is given by

$$H = -\sum_x (J_1 s + J_2 s s' s'') \tag{4.1}$$

i.e. an external field J_1 and an alternate three-spin interaction. This model is self-dual (Merlini and Gruber 1972). Following the same procedure to that in § 3, we add a new term to (4.1), equal to the product of s times $ss's''$. Thus the new Hamiltonian is

$$H = -\sum_x (J_1 s + J_2 s s' s'' + J_3 s' s''). \tag{4.2}$$

The partition function is given by

$$Z = \sum_s \exp\left(\sum_x K_1 s + K_2 s s' s'' + K_3 s' s''\right) \tag{4.3}$$

where $K_i = J_i/kT$. Following the above procedure we obtain

$$Z = (2\varepsilon)^N \sum_{s_1,s_2} \prod_x \exp[-(K_2^* s_1 + K_3^* s_2 + K_1^* s_1' s_1'' s_1''' s_2' s_2'')]. \tag{4.4}$$

There is no constraint on the spin variables when an external field is present.

In the same way to that of § 3, one of the spins can be ruled out. By changing the variables according to $s_1 \to s_1 s_2$, the summand in (4.4) results

$$\prod_x \exp[-(K_2^* s_1 s_2 + K_3^* s_2 + K_1^* s_1' s_1'' s_1''' s_2''')]$$
$$= \prod_x \exp[-(K_2^* s_1''' s_2''' + K_3^* s_2''' + K_1^* s_1' s_1'' s_1''' s_2''')]. \tag{4.5}$$

Since s_2 is decoupled, we can sum independently over each s_2 spin. Therefore, using the star-triangle relation above we obtain

$$Z=(2R\varepsilon)^N\sum_{s_1}\exp\left(\sum_x M_1s_1+M_2s_1's_1''s_1'''+M_3s_1's_1''\right). \tag{4.6}$$

Besides, owing to the symmetry of the lattice, (4.6) can be written as

$$Z=(2R\varepsilon)^N\sum_{s}\exp\left(\sum_x M_1s+M_2ss's''+M_3s's''\right) \tag{4.7}$$

where we have changed $s's''s'''$ by $ss's''$.

Therefore we have proved the self-duality of this model. The comments made in the final part of § 3 are valid for this case too.

Another analogous case is treated in our previous paper (Giacomini 1985). There we consider a self-dual model defined on a square lattice whose Hamiltonian is (Merlini and Gruber 1972)

$$H=-\sum_x (J_1s+J_2ss's''s'''). \tag{4.8}$$

As in the above examples, a new term was added in such a way that the new Hamiltonian results

$$H=-\sum_x (J_1s+J_2ss's''s'''+J_3s's''s'''). \tag{4.9}$$

It can be proved, by the same procedure as in the last example, that the resulting model is self-dual.

5. Self-duality of two interacting Baxter–Wu models

The 'Ising-type' version of the Ashkin–Teller Hamiltonian is given by (Fan 1972)

$$H=-\sum_x J_1(s_1s_1'+s_1s_1'')+J_2(s_2s_2'+s_2s_2'')+J_3(s_1s_1's_2s_2'+s_1s_1''s_2s_2''). \tag{5.1}$$

When $J_3=0$ the model reduces to two decoupled Ising models. Therefore the model is self-dual in that case. The terms added when $J_3\neq 0$ are products of terms present in the decoupled Hamiltonian, associated to the same site x. Hence the Ashkin–Teller model is analogous to the models considered above, and in consequence its self-duality can be proved by using the procedure applied to the previous cases. A model closely related to the Ashkin–Teller one is the following

$$\begin{aligned}Z=\sum_{s_1,s_2}\exp[&K_1(s_1s_1's_1''+s_1's_1''s_1''')+K_2(s_2s_2's_2''+s_2's_2''s_2''')\\&+K_3(s_1s_1's_1''s_2s_2's_2''+s_1's_1''s_1'''s_2's_2''s_2''')].\end{aligned} \tag{5.2}$$

When $K_3=0$ the system reduces to two uncoupled Baxter–Wu models and, therefore, is self-dual and exactly soluble (Baxter and Wu 1973); when $K_3\neq 0$ the new terms are products of the former ones. Hence the complete model must be self-dual too, as in the Ashkin–Teller case. We will prove this in the following.

By linearising the partition function in the spin variables and introducing six variables to expand the resulting products, we obtain

$$Z=\varepsilon^{2N}\sum_{s_1,\dots,s_6}\prod_x \exp[-(K_1^*(s_1+s_2)+K_2^*(s_3+s_4)+K_3^*(s_5+s_6)]$$
$$\times(1+s_1's_1''s_1'''s_2s_2's_2''s_5's_5''s_5'''s_6s_6's_6'')(1+s_3's_3''s_3'''s_4s_4's_4''s_5's_5''s_5'''s_6s_6's_6''). \quad (5.3)$$

Owing to the presence of two spin variables in the original model, two constraints result.

Now we change the variables as follows

$$s_1\to s_1s_5, \qquad s_2\to s_2s_6, \qquad s_3\to s_3s_5, \qquad s_4\to s_4s_6$$

with this, the summand in (5.3) results

$$\prod_x \exp\{-[K_1^*(s_1s_5+s_2s_6)+K_2^*(s_3s_5+s_4s_6)+K_3^*(s_5+s_6)]\}$$
$$\times(1+s_1's_1''s_1'''s_2s_2's_2'')(1+s_3's_3''s_3'''s_4s_4's_4''). \quad (5.4)$$

In this way, the variables s_5 and s_6 result decoupled and can be summed up. Therefore, using the star-triangle relation again, we obtain

$$Z=(R\varepsilon)^{2N}\sum_{s_1,s_2,s_3,s_4}\prod_x \exp(M_1s_3+M_2s_1+M_3s_1s_3+M_1s_4+M_2s_2+M_3s_2s_4)$$
$$\times(1+s_1's_1''s_1'''s_2s_2's_2'')(1+s_3's_3''s_3'''s_4s_4's_4''). \quad (5.5)$$

The sets of spins s_1, s_2, s_3, s_4 are restricted by the two constraints

$$s_1's_1''s_1'''s_2s_2's_2''=1, \qquad s_3's_3''s_3'''s_4s_4's_4''=1. \quad (5.6)$$

These are the same constraints that would result in the proof of the self-duality of two uncoupled Baxter-Wu models. Therefore, the solution of (5.6), as has been seen in § 2, is given by

$$s_1=s_7s_7's_7'', \qquad s_2=s_7's_7''s_7''', \qquad s_3=s_8s_8's_8'', \qquad s_4=s_8's_8''s_8''' \quad (5.7)$$

where s_7 and s_8 are also Ising variables.

In consequence the partition function which results is

$$Z=(2R\varepsilon)^{2N}\sum_{s_7,s_8}\exp\Bigg(\sum_x M_1(s_8s_8's_8''+s_8's_8''s_8''')+M_2(s_7s_7's_7''+s_7's_7''s_7''')$$
$$+M_3(s_7s_7's_7''s_8s_8's_8''+s_7's_7''s_7'''s_8's_8''s_8''')\Bigg). \quad (5.8)$$

Therefore the model is self-dual, and the self-duality relation is the same that for the Ashkin-Teller model. Also, both models have the same symmetries between the parameters K_1, K_2 and K_3, i.e.

$$Z(K_1,K_2,K_3)=Z(K_1,K_3,K_2)=Z(K_3,K_2,K_1)=Z(K_2,K_1,K_3). \quad (5.9)$$

However, the symmetries of the Hamiltonians of both models are different. Nevertheless, it would be interesting to analyse whether this model presents, as in the Ashkin-Teller case, a non-universal behaviour.

This could be done by a perturbative expansion around $K_3=0$ (Barber 1980).

In general, we can say that, given a self-dual model, by the procedure applied in the previous cases, we can generate a new model, which is also self-dual. However, in some cases, it is not possible to obtain a new model by the application of this method. This is the case, for example, for the Ising model on a two-dimensional triangular lattice. The Hamiltonian of this model is

$$H = -\sum_x (J_1 ss' + J_2 ss'' + J_3 s's''). \tag{5.10}$$

This system is self-dual, but if we multiply any pair of terms associated with a point x, we merely obtain the third such term. Another example of this fact is given by the Baxter model, whose Hamiltonian, expressed in terms of Ising variables, is given by (Wu 1971)

$$H = -\sum_x (J_1 ss''' + J_2 s's'' + J_3 ss's''s'''). \tag{5.11}$$

This model is self-dual, but our procedure does not generate a new interaction term.

6. Dual model to the anisotropic square Ising model with the next-nearest-neighbour interactions

It is interesting to apply the procedure developed in the previous sections to models that are not self-dual, as for example, the square Ising model with next-nearest-neighbour interactions. This system is described by the partition function

$$Z = \sum_s \exp\left(\sum_x K_1 ss' + K_2 ss'' + K_3 s's'' + K_4 ss'''\right). \tag{6.1}$$

By using the previous procedure we obtain

$$Z = \varepsilon^N \left(\frac{\sinh(2K_4)}{2}\right)^{N/2} \sum_{s_1,s_2,s_3,s_4} \prod_x \exp[-(K_1^* s_1 + K_2^* s_2 + K_2^* s_3 + K_3^* s_4)]$$
$$\times (1 + s_1 s_1' s_2 s_2'' s_3' s_3'' s_4 s_4'''). \tag{6.2}$$

Changing the variables as follows

$$s_1 \to s_1 s_3 \qquad s_2 \to s_2 s_3$$

the summand in (6.2) becomes

$$\prod_x \exp[-(K_1^* s_1 s_3 + K_2^* s_2 s_3 + K_3^* s_3 + K_4^* s_4)](1 + s_1 s_1' s_2 s_2'' s_4 s_4'''). \tag{6.3}$$

Summing up the spin s_3 and using the star-triangle relation we obtain

$$Z = (R\varepsilon)^N \left(\frac{\sinh(2K_4)}{2}\right)^{N/2} \sum_{s_1,s_2,s_4} \exp\left(\sum_x M_1 s_2 + M_2 s_1 + M_3 s_1 s_2 + K_4^* s_4\right)$$
$$\times (1 + s_1 s_1' s_2 s_2'' s_4 s_4''') \tag{6.4}$$

with the aim of simplifying the constraint $s_1 s_1' s_2 s_2'' s_4 s_4''' = 1$ we make a new change of variables

$$s_1 \to s_1 s_4'' \qquad s_2 \to s_2 s_4.$$

Thus, the constraint results

$$s_1 s_1' s_2 s_2'' = 1 \tag{6.5}$$

i.e. the same as that for the simple Ising model; therefore we have

$$s_1 = ss'' \qquad s_2 = ss' \tag{6.6}$$

and the partition function becomes

$$Z = (2R\varepsilon)^N \left(\frac{\sinh(2K_4)}{2}\right)^{N/2} \sum_{s,s_4} \exp\left(\sum_x M_1 ss's_4 + M_2 ss''s_4'' + M_3 s's''s_4 s_4'' + K_4^* s_4\right). \tag{6.7}$$

In this expression the spin variable s_4 is not decoupled and therefore cannot be eliminated. Hence the model is not self-dual.

However the dual model (6.7) can be useful, in some cases, for obtaining information about (6.1). The four spin interaction terms appearing in (6.7) could be a possible explanation for the non-universal behaviour occurring in (6.1), such as that which occurs for the Ashkin–Teller and the Baxter models. A more symmetric form of (6.7) can be obtained with the change of variables

$$s \to s_1 \qquad s_4 \to s_2 s_1$$

which result in

$$Z = (2R\varepsilon)^N (\tfrac{1}{2}\sinh(2K_4))^{N/2} \sum_{s_1,s_2} \exp\left(\sum_x M_1 s_1' s_2 + M_2 s_1 s_2'' + M_3 s_1 s_1' s_2 s_2'' + K_4^* s_1 s_2\right). \tag{6.8}$$

7. Conclusions

We have developed a procedure to obtain, from a self-dual spin-$\frac{1}{2}$ lattice system, a new self-dual model. The new model contains the original Hamiltonian plus new interaction terms formed by the products of the original ones, although we have shown that, in some cases, no new model is obtained by this method.

The technique used here is even valid for arbitrary spin-$\frac{1}{2}$ lattice systems with site-dependent interaction and external fields. In particular we have introduced a model formed by two interacting Baxter–Wu models that resembles the properties of the Ashkin–Teller system. Both models are self-dual and present the same symmetries in the interaction parameters, and decouple in soluble models when $K_3 = 0$. However the symmetries of the Hamiltonians are different. It would be interesting to study the possibility of a non-universal behaviour in this model, such as occurs in the Ashkin–Teller case. Other related approaches to duality transformations can be found in Wegner (1973) and Savit (1980).

Finally, we would like to mention that investigations are now in progress in order to attempt the generalisation of these results to models whose variables are elements of the group $Z(N)$.

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